

ON THE BOUNDARY VALUE PROBLEMS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS OF SECOND ORDER*

BY

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It is the purpose of this paper to investigate the solutions of linear differential equations of second order which satisfy linear boundary conditions of a general form. In the first section theorems regarding the existence of solutions of the differential equation

$$y'' + p(x)y' + q(x)y = f(x)$$

under the boundary conditions

$$a_1y(x_1) + a_2y(x_2) + a_3y'(x_1) + a_4y'(x_2) = A,$$

$$b_1y(x_1) + b_2y(x_2) + b_3y'(x_1) + b_4y'(x_2) = B,$$

will be established, under the assumption that the relation

$$a_2b_4 - a_4b_2 = e^{\int_{x_1}^{x_2} p dx} (a_1b_3 - a_3b_1)$$

is satisfied. † These results will be derived from the general solution of the differential equation. In the following sections the differential equation

$$y'' + p(x)y' + [\lambda A(x) - B(x)]y = 0$$

will be considered, where λ is a parameter. The existence of functions satisfying the differential equation and the homogeneous form of the above boundary conditions for particular values of λ , i. e., the existence of normal functions for the problem, will be proved by means of certain minimum problems, based upon that used by WEBER ‡ whereby reasoning similar to that of DIRICHLET's prin-

* Presented to the Society at the Williamstown summer meeting, September 8, 1905. Received for publication February 26, 1906.

† The problem might be called in this case a self adjoint boundary problem. The nature of the problem would be considerably different if this relation were not satisfied.

‡ *Mathematische Annalen*, vol. 1 (1869), p. 1.

ciple was employed. By this method the properties of the normal functions as minimum functions are set in evidence, properties upon which the proof of oscillation theorems for the normal functions may be based. As an illustration an oscillation theorem for periodic solutions will be proved in the last section.

Special cases of the above boundary conditions have formed the subject of many investigations since the famous memoirs of STURM and LIOUVILLE.* PICARD† has applied the method of successive approximations to the case $y(x_1) = 0$, $y(x_2) = 0$, for the differential equation

$$y'' + \lambda A(x)y = 0,$$

where the function A does not change sign. The writer‡ has treated certain boundary conditions for this differential equation by reducing the problem to the solution of an integral equation by means of one-dimensional Green's functions, and applying a method to which the method of § 2 of the present paper is analogous. In particular the existence of periodic solutions was thus shown. These results were later reproduced by HOLMGREN,§ using a method based upon HILBERT's treatment of DIRICHLET's principle, and by TZITZÉICA,|| who used a generalization of PICARD's method. The existence of periodic solutions was also proved by BÔCHER,¶ the proof being based on STURM's results.

Important results have recently been obtained by HILBERT regarding the solution of integral equations with symmetrical *Kern*.** These results may however be applied to the boundary problem of the differential equation only in case the function $A(x)$ does not change sign, a restriction which will not be made in the present article. The expansion of a function in terms of normal functions, which is the result of greatest interest in HILBERT's work, has also been treated by DIXON†† for a general boundary condition in connection with a differential equation to which the one here considered may be reduced if $A(x)$ does not change sign.

§ 1. *The existence of solutions of the general problem.*

It is required to determine a solution of the differential equation ‡‡

* *Journal de Mathématiques*, vols. 1, 2 (1836, 1837). See BÔCHER, *Encyklopädie der mathematischen Wissenschaften*, II A 7a.

† See his *Traité d'analyse*, 3 (1896), p. 105 ff.

‡ *Mathematische Annalen*, vol. 58 (1904), p. 528.

§ *Arkiv för Matematik, Astronomi och Fysik*, vol. 1 (1904), p. 401.

|| *Comptes rendus*, vol. 140 (1905), pp. 223, 492.

¶ *Ibid.*, vol. 140 (1905), p. 928.

** *Nachrichten der K. Gesellschaft der Wissenschaften zu Göttingen*, 1904. See also WESTFALL, *Dissertation*, Göttingen, 1905, and SCHMIDT, *Dissertation*, Göttingen, 1905.

†† *Proceedings of the London Mathematical Society*, ser. 2, vol. 3 (1905), p. 83.

‡‡ The coefficients of the different equations considered are assumed to be continuous functions in the interval (x_1, x_2) .

$$(1) \quad y'' + p(x)y' + q(x)y = f(x)$$

which satisfies the boundary conditions

$$(2) \quad \begin{cases} L_a(y) \equiv a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = A, \\ L_b(y) \equiv b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = B, \end{cases}$$

where A, B, a_i, b_i are given constants. It will be assumed that the coefficients of the boundary conditions satisfy the relation

$$a_2 b_4 - a_4 b_2 = e^{\int_{x_1}^{x_2} p dx} (a_1 b_3 - a_3 b_1),$$

or, in writing

$$a_i b_k - a_k b_i = d_{ik}, \quad e^{\int_{x_1}^{x_2} p dx} = \pi,$$

that the equation holds:

$$(3) \quad d_{24} = \pi d_{13}.$$

The general solution of equation (1) is

$$(4) \quad y = c_1 u + c_2 v + u \int_{x_1}^x f P v dx + v \int_x^{x_2} f P u dx,$$

where

$$P = e^{\int_{x_1}^x p dx},$$

c_1 and c_2 are arbitrary constants, and u, v are linearly independent solutions of the differential equation

$$(1_0) \quad y'' + p(x)y' + q(x)y = 0,$$

and are connected by the relation

$$(5) \quad u'v - v'u = e^{-\int_{x_1}^x p dx} = \frac{1}{P}.$$

The constants c_1, c_2 are to be so determined that y satisfies the boundary conditions (2). Substituting (4) in (2), writing $v(x_1) = v_1, v'(x_1) = v'_1$, etc., and arranging terms, we have:

$$(6) \quad \begin{cases} c_1 L_a(u) + c_2 L_a(v) = A - (a_1 v_1 + a_3 v'_1) \int_{x_1}^{x_2} f P u dx - (a_2 u_2 + a_4 u'_2) \int_{x_1}^{x_2} f P v dx, \\ c_1 L_b(u) + c_2 L_b(v) = B - (b_1 v_1 + b_3 v'_1) \int_{x_1}^{x_2} f P u dx - (b_2 u_2 + b_4 u'_2) \int_{x_1}^{x_2} f P v dx. \end{cases}$$

If in place of u, v any other pair of linearly independent solutions

$$\begin{aligned} U &= d_1 u + e_1 v, \\ V &= d_2 u + e_2 v \end{aligned} \quad (d_1 e_2 - e_1 d_2 \neq 0)$$

is used, the determinant $\Delta(u, v)$ of the first members of (6) will be replaced by

$$\Delta(U, V) = (d_1 e_2 - e_1 d_2) \Delta(u, v).$$

Then the following theorem holds;

If $\Delta \neq 0$ there exists a unique solution of the differential equation (1) under the boundary condition (2). There exists a solution of the homogeneous problem, i. e., a solution not identically zero of the differential equation (1₀) under the boundary conditions

$$(2_0) \quad \begin{cases} L_a(y) = 0, \\ L_b(y) = 0, \end{cases}$$

when and only when $\Delta = 0$.

Certain conditions must be satisfied by $A, B, f(x)$ in order that a solution of the non-homogeneous problem may exist when $\Delta = 0$. These conditions will be different according as the minors of Δ are or are not zero. The discussion of these two cases will be given separately.

Case I. The minors of Δ are not all zero.

In this case but one solution of the homogeneous problem exists, except for an arbitrary constant factor. Let u be this solution. Then v will not be a solution, i. e., $L_a(v)$ and $L_b(v)$ are not both zero. Since u satisfies (2₀) it will also satisfy the equations:

$$(7_1) \quad b_1 L_a(u) - a_1 L_b(u) \equiv d_{21} u(x_2) + d_{31} u'(x_1) + d_{41} u'(x_2) = 0,$$

$$(7_2) \quad b_2 L_a(u) - a_2 L_b(u) \equiv d_{12} u(x_1) + d_{32} u'(x_1) + \pi d_{31} u'(x_2) = 0,$$

$$(7_3) \quad b_3 L_a(u) - a_3 L_b(u) \equiv d_{13} u(x_1) + d_{23} u(x_2) + d_{43} u'(x_2) = 0,$$

$$(7_4) \quad b_4 L_a(u) - b_4 L_b(u) \equiv d_{14} u(x_1) + \pi d_{13} u(x_2) + d_{34} u'(x_1) = 0.$$

The necessary and sufficient condition for the existence of a solution of (1) and (2) is from (6):

$$\begin{aligned} L_a(v) \left\{ B - (b_1 v_1 + b_3 v'_1) \int_{x_1}^{x_2} f P u dx - (b_2 u_2 + b_4 u'_2) \int_{x_1}^{x_2} f P v dx \right\} \\ - L_b(v) \left\{ A - (a_1 v_1 + a_3 v'_1) \int_{x_1}^{x_2} f P u dx - (a_2 u_2 + a_4 u'_2) \int_{x_1}^{x_2} f P v dx \right\} = 0. \end{aligned}$$

Since u satisfies the boundary condition (2₀) the coefficient of the integral involving v takes the form:

$$\begin{aligned}
 & (a_1 v_1 + a_3 v'_1)(b_1 u_1 + b_3 u'_1) - (a_2 v_2 + a_4 v'_2)(b_2 u_2 + b_4 u'_2) \\
 & - (b_1 v_1 + b_3 v'_1)(a_1 u_1 + a_3 u'_1) + (b_2 v_2 + b_4 v'_2)(a_2 u_2 + a_4 u'_2) \\
 & = d_{13}(v_1 u'_1 - u_1 v'_1) - d_{24}(v_2 u'_2 - u_2 v'_2).
 \end{aligned}$$

From (3) and (5) this reduces to

$$d_{13} - d_{24} e^{-\int_{x_1}^{x_2} p dx} = d_{13} - \frac{d_{24}}{\pi} = 0,$$

and the above condition becomes :

$$\begin{aligned}
 (8) \quad & L_a(v) \left\{ B - (b_1 v_1 + b_3 v'_1) \int_{x_1}^{x_2} f P u dx \right\} \\
 & - L_b(v) \left\{ A - (a_1 v_1 + a_3 v'_1) \int_{x_1}^{x_2} f P u dx \right\} = 0.
 \end{aligned}$$

Now it may be shown by means of equations (3), (5) and (7) that the following equations hold :

$$(9_1) \quad L_a(v)(b_2 u'_1 + \pi b_1 u'_2) - L_b(v)(a_2 u'_1 + \pi a_1 u'_2) = 0,$$

$$(9_2) \quad L_a(v)(b_4 u_1 + \pi b_3 u_2) - L_b(v)(a_4 u_1 - \pi a_3 u_2) = 0,$$

$$(9_3) \quad L_a(v)(\pi b_3 u'_2 - b_2 u_1) - L_b(v)(\pi a_3 u'_2 - a_2 u_1) = 0,$$

$$(9_4) \quad L_a(v)(b_4 u'_1 - \pi b_1 u_2) - L_b(v)(a_4 u'_1 - \pi a_1 u_2) = 0.$$

In fact, expanding the left member of (9₁) and applying (3), (5) and (7), we have,

$$\begin{aligned}
 & d_{12} v_1 u'_1 + \pi d_{21} v_2 u'_2 + d_{32} u'_1 v'_1 + \pi d_{31} v'_1 u'_2 + d_{42} u'_1 v'_2 + \pi d_{41} u'_2 v'_2 \\
 & = d_{12} + v'_1(d_{12} u_1 + d_{32} u'_1 + \pi d_{31} u'_2) + d_{21} \\
 & + \pi v'_2(d_{21} u_2 + d_{31} u'_1 + d_{41} u'_2) = 0.
 \end{aligned}$$

The other equations of (9) are proved in a similar manner. Now $L_a(v)$ and $L_b(v)$ are not both zero, and hence the equation

$$\begin{aligned}
 (10) \quad & (a_2 u'_1 + \pi a_1 u'_2) \left\{ B - (b_1 v_1 + b_3 v'_1) \int_{x_1}^{x_2} f P u dx \right\} \\
 & - (b_2 u'_1 + \pi b_1 u'_2) \left\{ A - (a_1 v'_1 + a_3 v'_1) \int_{x_1}^{x_2} f P u dx \right\} = 0
 \end{aligned}$$

is a result of (8) and (9₁). Furthermore, (8) is a result of (9₁) and (10) unless

$$a_2 u'_1 + \pi a_1 u'_2 = 0, \quad b_2 u'_1 + \pi b_1 u'_2 = 0.$$

These equations can hold only when either $d_{12} = 0$ or $u'_1 = 0$, $u'_2 = 0$. But if $u'_1 = 0$, $u'_2 = 0$ then $u_1 \neq 0$, $u_2 \neq 0$, for a solution of (1₀) can not vanish together with its derivative, and it follows from (7₁) that $d_{12} = 0$. Therefore equation (10) is equivalent to (8) unless $d_{12} = 0$. If we expand the terms of (10) we have

$$(a_2 B - Ab_2)u'_1 + \pi(a_1 B - Ab_1)u'_2 - (d_{21}v_1u'_1 + d_{23}v'_1u'_1 + \pi d_{13}v'_1u'_2) \int_{x_1}^{x_2} fPudx = 0,$$

or, on account of (5) and (7₂),

$$(a_2 B - Ab_2)u'_1 + \pi(a_1 B - Ab_1)u'_2 + d_{12} \int_{x_1}^{x_2} fPudx = 0,$$

an equation which does not contain v , and which is equivalent to (8) except when $d_{12} = 0$. Three similar equations may be derived from (8) by using (9₂), (9₃) and (9₄). We have, replacing π and P by their values, the four equations,

$$(11) \left\{ \begin{aligned} & (a_2 B - Ab_2)u'_1 + e^{\int_{x_1}^{x_2} p^{dx}} (a_1 B - Ab_1)u'_2 + d_{12} \int_{x_1}^{x_2} f e^{\int_{x_1}^x p^{dx}} u dx = 0, \\ & (a_4 B - Ab_4)u_1 + e^{\int_{x_1}^{x_2} p^{dx}} (a_3 B - Ab_3)u_2 + d_{43} \int_{x_1}^{x_2} f e^{\int_{x_1}^x p^{dx}} u dx = 0, \\ & (a_2 B - Ab_2)u_1 - e^{\int_{x_1}^{x_2} p^{dx}} (a_3 B - Ab_3)u'_2 + d_{23} \int_{x_1}^{x_2} f e^{\int_{x_1}^x p^{dx}} u dx = 0 \\ & (a_4 B - Ab_4)u'_1 - e^{\int_{x_1}^{x_2} p^{dx}} (a_1 B - Ab_1)u_2 + d_{14} \int_{x_1}^{x_2} f e^{\int_{x_1}^x p^{dx}} u dx = 0, \end{aligned} \right.$$

each of which is equivalent to (8) unless the determinant d_{ik} which occurs in it is zero. Since from equation (3) d_{13} and d_{24} vanish together it is easily seen that all the determinants d_{12} , d_{43} , d_{23} , d_{14} cannot be zero unless all determinants d_{ik} are zero, a case to be excluded, since the boundary conditions (2) would be either incompatible or dependent. Therefore:

The necessary and sufficient condition for the existence of a solution of the differential equation

$$(1) \quad y'' + py' + qy = f,$$

under the boundary conditions

$$(2) \quad \begin{cases} L_a(y) \equiv a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = A, \\ L_b(y) \equiv b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = B, \end{cases}$$

when one solution, u , of the homogeneous problem

$$y'' + py' + qy = 0, \quad L_a(y) = 0, \quad L_b(y) = 0,$$

exists, but no other solution linearly independent of u , is that A , B , and $f(x)$ satisfy one of the equations (11) with non-vanishing determinant d_{ik} . If $A=0$, $B=0$ the condition is

$$(12) \quad \int_{x_1}^{x_2} f e^{\int_{x_1}^x p dx} u dx = 0.$$

If this condition is satisfied there exists a single infinity of solutions of (1) and (2) of the form

$$y = \bar{y} + cu,$$

where \bar{y} is any one such solution and c is an arbitrary constant.

CASE II. All minors of Δ are zero.

In this case the functions u and v are solutions of the homogeneous problem. The determinant d_{13} cannot be zero. For if $d_{13} = 0$ then $d_{24} = 0$ and since v satisfies the same boundary conditions as u it follows from (7₁) that

$$d_{21}u(x_2) + d_{41}u'(x_2) = 0,$$

$$d_{21}v(x_2) + d_{41}v'(x_2) = 0,$$

and therefore $d_{12} = 0$, $d_{14} = 0$, for the determinant $u(x_2)v'(x_2) - v(x_2)u'(x_2)$ is not zero. From equation (7₃) it follows in the same way that $d_{23} = 0$, $d_{34} = 0$ in case $d_{13} = 0$, that is, all determinants d_{ik} are zero if d_{13} is zero. We may therefore assume that d_{13} is not zero.

The necessary and sufficient conditions for the existence of a solution of equations (1) and (2) are seen from (6) and the boundary conditions for u and v to be

$$A - (a_1v_1 + a_3v'_1) \int_{x_1}^{x_2} fPudx + (a_1u + a_3u'_1) \int_{x_1}^{x_2} fPvdx = 0,$$

$$B - (b_1v_1 + b_3v'_1) \int_{x_1}^{x_2} fPudx + (b_1u + b_3u'_1) \int_{x_1}^{x_2} fPvdx = 0,$$

Eliminating each integral in turn we have, on account of (5),

$$(13) \quad \begin{cases} (a_1B - Ab_1)u_1 + (a_3B - Ab_3)u'_1 + d_{13} \int_{x_1}^{x_2} f e^{\int_{x_1}^x p dx} u dx = 0, \\ (a_1B - Ab_1)v_1 + (a_3B - Ab_3)v'_1 + d_{13} \int_{x_1}^{x_2} f e^{\int_{x_1}^x p dx} v dx = 0. \end{cases}$$

Since d_{13} can not be zero in this case, these equations are equivalent to the preceding pair, and hence:

The equations (13) are the necessary and sufficient conditions for the existence of a solution of the differential equation (1) under the boundary condition (2) in case two linearly independent solutions u, v of the homogeneous problem exist. If $A = 0, B = 0$ the conditions are

$$(14) \quad \int_{x_1}^{x_2} f e^{\int_{x_1}^{x_2} p dx} n dx = 0, \quad \int_{x_1}^{x_2} f e^{\int_{x_1}^{x_2} p dx} v dx = 0.$$

If these conditions are satisfied there exists a doubly infinite set of solutions of (1) and (2) of the form

$$y = \bar{y} + cu + dv,$$

where \bar{y} is any one such solution and c, d are arbitrary constants.

§ 2. The existence of the first normal functions.

There exists a solution of the homogeneous problem when and only when the determinant Δ of equations (6) is zero. If the coefficients of the differential equation contain a parameter λ , then Δ is a function of λ , and λ must have a value which is a zero of $\Delta(\lambda)$ in order that a solution of the required type exists. These zeros λ_i of $\Delta(\lambda)$ will be called *normal parameter values*, and the corresponding solutions y_i of the homogeneous problem *normal functions*. We shall consider the case that q is a linear function of λ . The problem to be treated is to prove the existence and investigate the properties of the normal parameter values and normal functions for the differential equation

$$(1') \quad y'' + p(x)y' + [\lambda A(x) - B(x)]y = 0$$

under the boundary conditions

$$(2_0) \quad \begin{cases} L_a(y) \equiv a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = 0, \\ L_b(y) \equiv b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = 0. \end{cases}$$

It will be assumed that the following conditions are satisfied:

$$(3) \quad d_{24} = \pi d_{13} \quad \left(\pi = e^{\int_{x_1}^{x_2} p dx} \right),$$

$$(15) \quad B(x) \geq 0 \quad (x_1 \leq x \leq x_2),$$

(16) all the determinants $d_{12}, d_{14}, d_{23}, d_{43}$ that are not zero have the same sign.

The following statement is a result of condition (16):

LEMMA. The values of the expression

$$- [yy' e^{\int_{x_1}^x p dx}]_{x_1}^{x_2} \equiv y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)$$

formed for all functions which satisfy (2₀) are positive or zero.

Any function y which satisfies (2_0) also satisfies the equations

$$(17_1) \quad d_{21}y(x_2) + d_{31}y'(x_1) + d_{41}y'(x_2) = 0,$$

$$(17_2) \quad d_{12}y(x_1) + d_{32}y'(x_1) + \pi d_{31}y'(x_2) = 0,$$

$$(17_3) \quad d_{13}y(x_1) + d_{23}y(x_2) + d_{43}y'(x_2) = 0,$$

$$(17_4) \quad d_{14}y(x_1) + \pi d_{13}y(x_2) + d_{34}y'(x_1) = 0.$$

Hence

$$(18_1) \quad d_{12}[y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)] = d_{23}[y'(x_1)]^2 + 2\pi d_{13}y'(x_1)y'(x_2) + \pi d_{14}[y'(x_2)]^2,$$

$$(18_2) \quad d_{43}[y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)] = d_{14}[y(x_1)]^2 + 2\pi d_{13}y(x_1)y(x_2) + \pi d_{23}[y(x_2)]^2,$$

$$(18_3) \quad d_{23}[y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)] = d_{12}[y(x_1)]^2 + \pi d_{43}[y'(x_2)]^2,$$

$$(18_4) \quad d_{14}[y(x_1)y'(x_1) - \pi y(x_2)y'(x_2)] = d_{43}[y'(x_1)]^2 + \pi d_{12}[y(x_2)]^2.$$

The discriminants of the right members of (18_1) and (18_2) are

$$\pi(d_{14}d_{23} - \pi d_{13}^2) = \pi(d_{14}d_{23} - d_{13}d_{24}).$$

The second member when expanded is seen to be equal to

$$\pi d_{12}d_{43}.$$

Since the determinants d_{ik} which occur in the first members of (18) cannot all be zero the truth of the above lemma is seen at once.

Consider now the following minimum problem:

*Among all functions $y(x)$ which satisfy the boundary conditions (2_0) and the equation **

$$(19) \quad \int_{x_1}^{x_2} APy^2 dx = 1 \quad \left(P = e^{\int_{x_1}^x p dx} \right),$$

and are continuous together with their first and second derivatives in the interval (x_1, x_2) , that one is required which gives the least value to the expression

$$J = \int_{x_1}^{x_2} (y'^2 + By^2) P dx - [yy'P]_{x_1}^{x_2}.$$

The values of J are all positive or zero, on account of conditions (15) and the lemma proved by means of condition (16) . Hence there exists a definite *lower limit* λ_0 of the values of J under the above conditions. It will be proved that

* It is here assumed that A is not everywhere negative in (x_1, x_2) . This is no restriction on the problem since A occurs in the equation $(1')$ multiplied by the parameter λ .

this value λ_0 is a normal parameter value for the differential equation (1') under the boundary conditions (2₀), and that the corresponding normal function is the solution of the minimum problem. If λ_0 were not a normal parameter value there would exist, according to the theorems of § 1, a solution of the differential equation

$$y'' + py' + (\lambda_0 A - B)y = f$$

under the boundary conditions (2₀), whatever be f . It will be shown that f could be so chosen that the solution of the equation would satisfy all conditions of the minimum problem, and would give to J a value less than the lower limit of all such values, λ_0 . The value λ_0 must therefore be a normal parameter value.

Let

$$u_1, u_2, u_3, \dots$$

be an infinite series of approximating functions for the minimum problem, i. e., an infinite series of functions u_h which satisfy the equation

$$(20) \quad \lim_{h \rightarrow \infty} J(u_h) = \lambda_0,$$

and all the conditions of the minimum problem.

LEMMA. *The approximating functions u_h can be so chosen that there exists a fixed number G such that for all h*

$$(21) \quad |u_h(x)| < G \quad (x_1 \leq x \leq x_2)$$

From the inequalities,

$$\begin{aligned} \left| \int_{x_1}^x u'_h dx \right| &\leq \int_{x_1}^x |u'_h| dx < \int_{x_1}^x (u_h'^2 + 1) dx, \\ \int_{x_1}^x u'_h dx &\leq \frac{1}{m} \int_{x_1}^x P u_h'^2 dx < \frac{J(u_h)}{m}, \end{aligned}$$

where m is the minimum of $P = e^{\int_{x_1}^x p dx}$ in the interval (x_1, x_2) , it follows that

$$|u_h(x) - u_h(x_1)| < \frac{\lambda_0}{m} + (x_2 - x_1) + \epsilon_h,$$

where $\lim_{h \rightarrow \infty} \epsilon_h = 0$. The functions u_h therefore have the form

$$u_h(x) = c_h + \gamma_h(x),$$

where c_h are constants and γ_h a series of functions such that the maxima of their absolute values remain under a fixed limit. From the equation

$$\int_{x_1}^{x_2} APu_h^2 dx = c_h^2 \int_{x_1}^{x_2} AP dx + 2c_h \int_{x_1}^{x_2} AP\gamma_h dx + \int_{x_1}^{x_2} AP\gamma_h^2 dx = 1,$$

it follows that the quantities c_h all lie between finite limits unless

$$\int_{x_1}^{x_2} AP dx = 0, \quad \lim_{h \rightarrow \infty} \int_{x_1}^{x_2} A\gamma_h dx = 0,$$

and consequently

$$(22) \quad \lim_{h \rightarrow \infty} \int_{x_1}^{x_2} A\gamma_h^2 dx = 1.$$

From equation (18₄), which is satisfied by the functions u_h , we have

$$d_{43} [u'_h(x_1)]^2 + \pi d_{12} [u_h(x_2)]^2 = -d_{14} [u_h u'_h P]_{x_1}^{x_2} \leq d_{12} J(u_h).$$

Hence, if $|c_h|$ increase beyond limit, $d_{12} = 0$ and either $d_{43} = 0$, or the values $u'_h(x_1)$ remain between finite limits. Similarly from (18₃), either $d_{43} = 0$, or $u'_h(x_2)$ remain between finite limits, and hence, from (17₃) and (17₄),

$$d_{13} + d_{23} = 0, \quad d_{14} + \pi d_{13} = 0.$$

Therefore either $d_{13} \neq 0$ or $d_{34} \neq 0$, for if both are zero then all determinants d_{ik} are zero. Then the boundary conditions (2₀) are equivalent to either (17₁) and (17₃), or to (17₃) and (17₄). Hence, if $|c_h|$ increase beyond limit, the boundary conditions, since d_{12} must be zero, are equivalent to

$$y'(x_1) - \pi y'(x_2) = 0,$$

$$y(x_1) - y(x_2) = 0,$$

if $d_{34} = 0$, or to

$$d_{13} [y(x_1) - y(x_2)] + d_{43} y'(x_2) = 0,$$

$$\pi d_{13} [y(x_1) - y(x_2)] + d_{43} y'(x_1) = 0,$$

if $d_{34} \neq 0$. [In either case the boundary conditions are satisfied by $y = \text{const.}$, and hence by $\gamma_h = u_h - c_h$], and we have also

$$-[\gamma_h \gamma'_h P]_{x_1}^{x_2} = -[u_h u'_h P]_{x_1}^{x_2} + c_h [\pi u'_h(x_2) - u'_h(x_1)] = -[u_h u'_h P]_{x_1}^{x_2}.$$

Therefore

$$J(\gamma_h) = J(u_h).$$

Let

$$\delta_h = \int_{x_1}^{x_2} AP\gamma_h^2 dx.$$

From equation (22)

$$\lim_{h \rightarrow \infty} \delta_h = 1.$$

Therefore if $|c_h|$ increase without limit, i. e., if the functions u_h do not satisfy condition (21) there exists a series of functions

$$\bar{u}_h(x) = \frac{u_h(x) - c_h}{\sqrt{|\delta_h|}}$$

which satisfy (21), and since they satisfy the equations

$$\begin{aligned} L_a(\bar{u}_h) &= 0, & L_b(\bar{u}_h) &= 0, \\ \int_{x_1}^{x_2} AP\bar{u}_h^2 dx &= \frac{1}{\delta_h} \int_{x_1}^{x_2} AP\gamma_h^2 dx = 1, \\ \lim_{h \rightarrow \infty} J(\bar{u}_h) &= \lim_{h \rightarrow \infty} \frac{1}{\delta_h} J(u_h) = \lambda_0, \end{aligned}$$

they form a series of approximating functions for the minimum problem. The lemma stated above is therefore proved.

Define an infinite series of functions f_h by the equations

$$(23) \quad (u'_h P)' + (\lambda_0 A - B) P u_h = f_h.$$

Multiplying these equations by u_h and integrating from x_1 to x_2 we have, on account of the condition (19),

$$[u_h u'_h P]_{x_1}^{x_2} - \int_{x_1}^{x_2} (u_h'^2 + B u_h^2) P dx + \lambda_0 = \int_{x_1}^{x_2} f_h u_h dx.$$

Hence, on account of (20),

$$(24) \quad \lim_{h \rightarrow \infty} \int_{x_1}^{x_2} f_h u_h dx = 0.$$

Under the assumption that λ_0 is not a normal parameter value there exists for each h a solution of the differential equation

$$U_h'' + p U_h' + (\lambda_0 A - B) U_h = A u_h,$$

or, multiplying by $P = e^{\int_{x_1}^x p dx}$, of the equation

$$(25) \quad (U_h' P)' + (\lambda_0 A - B) P U_h = P A u_h,$$

under the boundary conditions (2₀). Consider the functions

$$v_h = u_h + c U_h$$

where c is a constant. These functions satisfy the boundary conditions (2₀) and the equation

$$(v_h' P)' + (\lambda_0 A - B) P v_h = f_h + c P A u_h.$$

Multiplying this equation by $v_\lambda = u_\lambda + c U_\lambda$ and integrating from x_1 to x_2 the following equation results:

$$(26) \quad \begin{aligned} [v_\lambda v'_\lambda P]_{x_1}^{x_2} - \int_{x_1}^{x_2} (v_\lambda'^2 + B v_\lambda^2) P dx + \lambda_0 \int_{x_1}^{x_2} A P v_\lambda^2 dx \\ = \int_{x_1}^{x_2} f_\lambda u_\lambda dx + c \int_{x_1}^{x_2} (f_\lambda U_\lambda + P A u_\lambda^2) dx + c^2 \int_{x_1}^{x_2} P A u_\lambda U_\lambda dx. \end{aligned}$$

Now from equations (23) and (25) it follows that

$$\begin{aligned} \int_{x_1}^{x_2} (f_\lambda U_\lambda - P A u_\lambda^2) dx = \int_{x_1}^{x_2} \{ (u'_\lambda P)' U_\lambda - (U'_\lambda P) u_\lambda \} dx \\ = [(u'_\lambda U_\lambda - U'_\lambda u_\lambda) P]_{x_1}^{x_2}. \end{aligned}$$

The functions u_λ and U_λ satisfy the boundary conditions (2₀) and hence equations (17). Since $P(x_1) = 1$, $P(x_2) = \pi$, it follows from (17₁) and (17₂) that

$$\begin{aligned} d_{12} [(u'_\lambda U_\lambda - U'_\lambda u_\lambda) P]_{x_1}^{x_2} = \pi u'_\lambda(x_2) [d_{31} U'_\lambda(x_1) + d_{41} U'_\lambda(x_2)] \\ - \pi U'_\lambda(x_2) [d_{31} u'_\lambda(x_1) + d_{41} u'_\lambda(x_2)] - u'_\lambda(x_1) [d_{23} U'_\lambda(x_1) + \pi d_{13} U'_\lambda(x_2)] \\ + U'_\lambda(x_1) [d_{23} u'_\lambda(x_1) + \pi d_{13} u'_\lambda(x_2)] = 0. \end{aligned}$$

Hence

$$[(u'_\lambda U_\lambda - U'_\lambda u_\lambda) P]_{x_1}^{x_2} = 0$$

unless $d_{12} = 0$. By using other pairs of equations (17) it is seen that this equation must hold in any case, since all the determinants d_{ik} cannot be zero. Therefore

$$\int_{x_1}^{x_2} (f_\lambda u_\lambda - P A u_\lambda^2) dx = 0.$$

Since

$$\int_{x_1}^{x_2} P A u_\lambda^2 dx = 1,$$

equation (26) becomes

$$(27) \quad \lambda_0 \int_{x_1}^{x_2} P A v_\lambda^2 dx - J(v_\lambda) = \int_{x_1}^{x_2} f_\lambda u_\lambda dx + 2c + c^2 \int_{x_1}^{x_2} P A u_\lambda U_\lambda dx.$$

Now from equation (4)

$$U_\lambda = c_\lambda u + d_\lambda v + u \int_{x_1}^x P A u_\lambda v dx + v \int_x^{x_2} P A u_\lambda u dx,$$

where u, v are the linearly independent solutions of the equation

$$y'' + py' + (\lambda A - B)y = 0$$

and c_h, d_h are constants whose values are so determined that U_h satisfies the boundary conditions (2₀). Referring to equations (6) one sees that c_h and d_h are linear combinations of the two integrals

$$\int_{x_1}^{x_2} PAu_h u dx, \quad \int_{x_1}^{x_2} PAu_h v dx,$$

the coefficients being constants independent of h . Therefore since the functions u_h satisfy condition (20) it follows that the functions U_h satisfy a similar condition, and that the quantities

$$\left| \int_{x_1}^{x_2} PAu_h U_h dx \right|,$$

which occur in (27) remain for all h under some fixed limit B . Then from (27)

$$\lambda_0 \int_{x_1}^{x_2} PA v_h^2 dx - J(v_h) > \int_{x_1}^{x_2} f_h v_h dx + 2c - c^2 B.$$

Let c be chosen as a positive number so small that

$$2c - c^2 B > 0.$$

Write $2c - c^2 B = \delta$. It follows from (24) that h may be taken so large that

$$\int_{x_1}^{x_2} f_h u_h dx < \delta.$$

Then for this value of h we have

$$(28) \quad \lambda_0 \int_{y_1}^{x_2} PA v_h^2 dx - J(v_h) > 0.$$

Therefore

$$\int_{x_1}^{x_2} PA v_h^2 dx > 0,$$

and

$$\alpha = \left\{ \int_{x_1}^{x_2} PA v_h^2 dx \right\}^{\frac{1}{2}}$$

is a real number. Let

$$y = \frac{v_h}{\alpha}.$$

This function satisfies all conditions of the minimum problem. It must therefore give to J a value less than or equal to λ_0 , the lower limit of the values of J under the conditions of the problem. But dividing (28) by α^2 we have

$$\lambda_0 - \frac{1}{\alpha^2} J(v_h) = \lambda_0 - J(y) > 0,$$

which is impossible. The assumption that λ_0 is not a normal parameter value therefore leads to a contradiction. Hence the lower limit λ_0 is a normal parameter value, and there exists a normal function y_0 , not identically zero, which satisfies the differential equation

$$(29) \quad y_0'' + py_0' + (\lambda_0 A - B)y_0 = 0$$

and the boundary conditions (2₀).

Multiplying equation (29) by Py_0 and integrating from x_1 to x_2 , we derive the equation

$$(30) \quad \lambda_0 \int_{x_1}^{x_2} PAy_0^2 dx = J(y_0).$$

Since λ_0 is positive or zero the integral in the first member is positive unless $J(y_0) = 0$, i. e., unless y_0 is a constant. Hence the arbitrary constant factor of y_0 can be so determined that

$$(31) \quad \int_{x_1}^{x_2} PAy_0^2 dx = 1,$$

unless $y_0 = \text{const.}$ Even if $y_0 = \text{const.}$ this determination may be made, unless

$$(32) \quad \int_{x_1}^{x_2} PA dx = 0;$$

for if the integral is not zero it may be assumed without restriction to be positive, as the function A occurs in the differential equation multiplied by the parameter λ . Now equation (32) cannot hold if $y_0 = \text{const.}$, since if (32) is satisfied λ_0 must be greater than zero. In fact, since the approximating functions u_h satisfy the equation

$$\int_{x_1}^{x_2} PAu_h^2 dx = 1,$$

it follows that if (32) holds, each function u_h must assume for some two values $x = x_h, x = \bar{x}_h$ two values $u_h(x_h), u_h(\bar{x}_h)$ which differ by an amount equal to δ , a fixed positive number independent of h . Let Y_h represent the ordinate of the straight line connecting the two points $[x_h, u_h(x_h)]$ and $[\bar{x}_h, u_h(\bar{x}_h)]$. Then

$$|Y_h'| = \frac{\delta}{\bar{x}_h - x_h}, \quad Y_h'' = 0.$$

Now

$$J(u_h) \geq \int_{x_h}^{\bar{x}_h} u_h'^2 dx.$$

Write

$$\eta_h = Y_h - u_h.$$

Then since η_h vanishes at x_h and at \bar{x}_h , and $F'' = 0$,

$$\int_{x_h}^{\bar{x}_h} u_h'^2 dx = \int_{x_h}^{\bar{x}_h} (F'_h - \eta_h')^2 dx = \frac{\delta^2}{\bar{x}_h - x_h} + \int_{x_h}^{\bar{x}_h} \eta_h'^2 dx.$$

Therefore, for all values of h ,

$$J(u_h) > \frac{\delta^2}{x_2 - x_1},$$

and λ_0 cannot be zero.

It may therefore be assumed without restriction that the normal function y_0 satisfies equation (31). Hence from (30) y_0 gives to J the value λ_0 . *The normal function y_0 is therefore the solution of the minimum problem.*

§ 3. The existence of an infinite series of normal functions.

We proceed by complete induction. Suppose that there exist n positive or zero normal parameter values

$$\lambda_0, \lambda_1, \dots, \lambda_{n-1} \quad (\lambda_{i+1} \geq \lambda_i)$$

and n linearly independent normal functions

$$y_0, y_1, \dots, y_{n-1}$$

satisfying the differential equations

$$y_i'' + py_i' + (\lambda_i A - B)y_i = 0$$

and the boundary conditions (2₀). At most two of the values λ_i can coincide, since the functions y_i are assumed to be linearly independent. Multiplying the differential equation by Py_i and integrating from x_1 to x_2 , we have

$$\lambda_i \int_{x_1}^{x_2} P A y_i^2 dx = J(y_i).$$

The arbitrary constant factor of y_i may therefore be so determined that

$$(33) \quad \int_{x_1}^{x_2} P A y_i^2 dx = 1,$$

except possibly when $J(y_i) = 0$, since λ_i and $J(y_i)$ are positive or zero. If $J(y_i) = 0$ then y_i is a constant and $\lambda_i = 0$; hence $y_i = y_0$, and it has been shown above that y_0 satisfies (31). Furthermore, from the equations

$$(Py_i')' + (\lambda_i A - B)y_i = 0,$$

$$(Py_k')' + (\lambda_k A - B)y_k = 0,$$

follows

$$(\lambda_i - \lambda_k) \int_{x_1}^{x_2} P A y_i y_k dx = [P(y'_i y_i - y'_k y_k)]_{x_1}^{x_2}.$$

Since y_i and y_k satisfy the boundary conditions (2₀), the right member is zero,* and hence if $\lambda_i \neq \lambda_k$,

$$(34) \quad \int_{x_1}^{x_2} P A y_i y_k dx = 0.$$

If $\lambda_i = \lambda_k$ then the functions y_i and y_k are not uniquely determined. The function y_k has the form

$$y_k = Y_k + c y_i,$$

where c is an arbitrary constant. The value of c may be determined so that

$$\int_{x_1}^{x_2} P A y_i y_k dx = \int_{x_1}^{x_2} P A y_i Y_k dx + c = 0.$$

It may therefore be assumed without restriction that the n normal functions y_i satisfy the equations

$$(33) \quad \int_{x_1}^{x_2} P A y_i^2 dx = 1$$

$$(34) \quad \int_{x_1}^{x_2} P A y_i y_k dx = 0 \quad (i \neq k).$$

To prove the existence of a normal function y_n linearly independent of y_0, y_1, \dots, y_{n-1} , consider the following minimum problem:

It is required to minimize the expression

$$J(u) = \int_{x_1}^{x_2} (y'^2 + B y^2) P dx - [P y y']_{x_1}^{x_2}$$

under the conditions

$$L_a(y) = 0, \quad L_b(y) = 0,$$

$$(35) \quad \int_{x_1}^{x_2} P A y^2 dx = 1,$$

$$(36) \quad \int_{x_1}^{x_2} P A y_i y dx = 0 \quad (i = 0, 1, 2, \dots, n-1).$$

Under the conditions (15), (16) there exists a finite lower limit λ_n of the values of J , and an infinite series of approximating functions

$$u_1, u_2, u_3, \dots$$

* The proof is the same as that given in the reduction of equation (26) to (27).

which satisfy all the conditions of the minimum problem and the equation

$$\lim_{h \rightarrow \infty} J(u_h) = \lambda_n.$$

Define as before an infinite series of functions f_h , by the equations

$$(Pu'_h)' + (\lambda_n A - B)Pu_h = f_h.$$

Under the assumption that for $\lambda = \lambda_n$ there exists no solution of (1') and (2₀) which is linearly independent of y_0, y_1, \dots, y_{n-1} , there must exist, for each value of h , a solution of the equation

$$(37) \quad U''_h + pU'_h + (\lambda_n A - B)U_h = Au_h$$

under the boundary conditions (2₀). For by the theorems of § 1 the equations

$$\int_{x_1}^{x_2} Ay_i u_h dx = 0, \quad \int_{x_1}^{x_2} Ay_k u_h dx = 0 \quad (h=1, 2, 3, \dots),$$

which are satisfied by all the functions u_h , are sufficient conditions for the existence of such solutions U_h in case $\lambda_n = \lambda_i = \lambda_k$; the first equation alone is the sufficient condition in case λ_n is equal to but one of the normal values λ_i ; and there is no condition necessary in case λ_n is not a normal value. The possibility that λ_n is a normal parameter value different from $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ is excluded by the assumption that for $\lambda = \lambda_n$ no solution of (1') and (2₀) exists which is linearly independent of y_0, y_1, \dots, y_{n-1} .

Since

$$\int_{x_1}^{x_2} PAy_i u_h dx = 0 \quad \left(\begin{array}{l} i=0, 1, \dots, n-1 \\ h=1, 2, 3, \dots \end{array} \right),$$

it follows from the equations

$$(Py'_i)' + (\lambda_i A - B)Py_i = 0,$$

$$(PU'_h)' + (\lambda_n A - B)PU_h = PAu_h,$$

that

$$(\lambda_n - \lambda_i) \int_{x_1}^{x_2} PAy_i U_h dx = [P(y_i U_h - U'_h y_i)]_{x_1}^{x_2}.$$

The right member is zero since y_i and U_h both satisfy the boundary conditions (2₀). Hence if $\lambda_n \neq \lambda_i$ the functions v_h satisfy the equations

$$\int_{x_1}^{x_2} Ay_i U_h dx = 0.$$

If $\lambda_n = \lambda_i$, then the functions U_h are not uniquely determined by (37) and (2₀), but have the form

$$U_h = \bar{U}_h + c_n y_i,$$

where c_n are arbitrary constants. These constants may be so determined that

$$\int_{x_1}^{x_2} P A y_i U_h dx = \int_{x_1}^{x_2} P A y_i \bar{U}_h dx + c_h = 0.$$

The functions U_h , and therefore the functions

$$v_h = u_h + c U_h,$$

where c is a constant, satisfy all the conditions of the minimum problem except

$$\int_{x_1}^{x_2} P A v_h^2 dx = 1.$$

The assumption, that for $\lambda = \lambda_n$ no normal function linearly independent of y_0, y_1, \dots, y_{n-1} exists, may now be shown to be false by the method used in the case of the first normal function. There exists $n + 1$ normal functions provided n exist, and the existence of a first has been proved. Hence an infinite series of normal functions y_n exists, and an infinite series of corresponding normal parameter values λ_n which are defined as the lower limits of the expression $J(y)$ under the conditions (2₀), (35), (36).

As has been shown above, it may be assumed without restriction that these functions satisfy the equations

$$\begin{aligned} \int_{x_1}^{x_2} P A y_n y_m dx &= 0 & (n \neq m), \\ \int_{x_1}^{x_2} P A y_n^2 dx &= 1. \end{aligned}$$

The function y_n therefore satisfies the conditions of the minimum problem used to define λ_n . Multiplying the equation

$$(P y_n')' + (\lambda_n A - B) P y_n = 0$$

by y_n and integrating from x_1 to x_2 , we have

$$J(y_n) = \lambda_n.$$

Hence y_n is the solution of the minimum problem. From the definition of λ_n as the lower limit of J under the conditions, it may be seen without difficulty that the series y_n contains all normal functions whose corresponding parameter value λ_n is positive or zero.

The values λ_n increase without limit, for they are the zeros of Δ , which is an integral transcendental function of λ .* If these zeros had a limiting value $\bar{\lambda}$ then Δ would vanish for $\lambda = \bar{\lambda}$ together with all its derivatives.

If A changes sign in the interval (x_1, x_2) the above method, with the single alteration that the equation

$$\int_{x_1}^{x_2} P A y^2 dy = 1$$

be replaced by

$$\int_{x_1}^{x_2} P A y^2 dx = -1,$$

shows the existence of an infinite series of negative normal parameter values λ_i and corresponding normal functions y_i . The results may be summarized in the following

THEOREM. *There exists an infinite series of normal parameter values λ_n and corresponding normal functions y_n for the differential equation*

$$(1') \quad y'' + p(x)y' + [\lambda A(x) - B(x)]y = 0$$

under the boundary conditions

$$(2_0) \quad \begin{cases} a_1 y(x_1) + a_2 y(x_2) + a_3 y'(x_1) + a_4 y'(x_2) = 0, \\ b_1 y(x_1) + b_2 y(x_2) + b_3 y'(x_1) + b_4 y'(x_2) = 0, \end{cases}$$

where, in writing $a_i b_k - a_k b_i = d_{ik}$,

$$e^{\int_{x_1}^{x_2} p dx} d_{13} = d_{24},$$

and where all the determinants $d_{12}, d_{14}, d_{23}, d_{43}$ which are not zero have the same sign.

If the function $A(x)$ is not negative in the interval (x_1, x_2) the values λ_i are not negative, and increase without limit with i . If $A(x)$ changes sign in the interval the values λ_i include an infinite series of positive quantities increasing without limit and an infinite series of negative quantities decreasing without limit.

The function y_n minimizes the expression

$$J(y) = \int_{x_1}^{x_2} (y'^2 + B y^2) P dx - [y y' P]_{x_1}^{x_2}$$

under the conditions

$$L_a(y) = 0, \quad L_b(y) = 0,$$

* See PICARD, *Traité d'analyse*, t. 3 (1896), p. 92.

$$\int_{x_1}^{x_2} P A y^2 dx = \pm 1,$$

$$\int_{x_1}^{x_2} P A y_i y dx = 0 \quad [i=0, \pm 1, \pm 2, \dots, \pm (n-1)],$$

and gives to J the value $\pm \lambda_n$, the upper signs being taken if λ_n is positive, the lower signs if λ_n is negative.

§ 4. An oscillation theorem for the periodic solutions.

If the coefficients of (1') are periodic functions with the period $x_2 - x_1$ then any solution of (1') under the boundary conditions

$$y(x_1) - y(x_2) = 0,$$

$$y'(x_1) - y'(x_2) = 0,$$

is likewise periodic with the period $x_2 - x_1$. In this case, since $d_{12} = 0$, $d_{14} = -1$, $d_{23} = -1$, $d_{43} = 0$, conditions (16) are satisfied. Since $d_{13} = 1$, $d_{24} = 1$ we must have

$$\pi = e^{\int_{x_1}^{x_2} p dx} = 1,$$

or

$$\int_{x_1}^{x_2} p dx = 0,$$

in order to satisfy condition (3). It will be assumed that this condition is satisfied. Then there exists an infinite series of periodic solutions y_n of (1') when the coefficients are periodic. We shall investigate the number of times y_n vanishes in the interval (x_1, x_2) under the assumption that

$$A(x) \geq 0 \quad (x_1 \leq x \leq x_2).$$

Let μ_n and u_n denote the normal parameter values and functions for the boundary condition

$$y(x_1) = 0, \quad y(x_2) = 0.$$

Then, under the assumption $A \geq 0$, according to STURM's theorems* the function u_n vanishes n times *within* the interval (x_1, x_2) . We shall determine the number of times the periodic function y_n vanishes in (x_1, x_2) by means of this fact.

It may be proved from the definition of the normal parameter values as minimal values that

$$\lambda_n \leq \mu_n \leq \lambda_{n+1}.$$

* See BÖCHER, *Encyklopädie der mathematischen Wissenschaften*, II A 7a.

To show that $\mu_n \leq \lambda_{n+1}$ it suffices to form a function w which satisfies the conditions

$$w(x_1) = 0, \quad w(x_2) = 0,$$

$$\int_{x_1}^{x_2} PAw^2 dx = 1, \quad \int_{x_1}^{x_2} PAu_i w dx = 0 \quad (i = 0, 1, 2, \dots, n-1),$$

and gives to

$$\int_{x_1}^{x_2} (w^2 + Bw^2) P dy$$

a value less than or equal to λ_{n+1} . Consider the function

$$w = c_0 y_0 + c_1 y_1 + \dots + c_{n+1} y_{n+1},$$

formed from the periodic solutions y_i . Since $w(x_1) = w(x_2)$ the above boundary condition will be satisfied if $w(x_1) = 0$. Now the $n+2$ constants c_i may always be so determined that they are not all zero, and that the $n+1$ equations

$$w(x_1) = \sum c_k y_k(x_1) = 0,$$

$$\int_{x_1}^{x_2} PAu_i w dx = \sum c_k \int_{x_1}^{x_2} PAu_i y_k dx = 0 \quad (i = 0, 1, \dots, n-1)$$

are satisfied, and furthermore so that

$$\int_{x_1}^{x_2} PAw^2 dx = \sum c_k^2 = 1.$$

Then w satisfies all the conditions of the minimum problem used in defining μ_n .

Multiplying the equation

$$(Py'_i)' + (\lambda_i A - B)Py_i = 0$$

by y_k , integrating from x_1 to x_2 and remembering the conditions which y_n satisfies, we have

$$-\int_{x_1}^{x_2} Py'_i y'_k dx - \int_{x_1}^{x_2} PBy_i y_k dx = 0.$$

Hence

$$\int_{x_1}^{x_2} (w^2 + Bw^2) P dx = \sum c_k^2 \int_{x_1}^{x_2} (y_k'^2 + By_k^2) P dx = \sum c_k^2 \lambda_k \leq \lambda_{n+1} \sum c_k^2 = \lambda_{n+1}.$$

Therefore

$$\mu_n \leq \lambda_{n+1}.$$

To prove the inequality

$$\lambda_n \leq \mu_n,$$

it is to be noticed that λ_n is the lower limit of the values of

$$J(y) = \int_{x_1}^{x_2} (y^2 + By^2) P dx$$

under the conditions

$$\int_{x_1}^{x_2} P A y^2 dx = 1, \quad \int_{x_1}^{x_2} P A y_i y dx = 0 \quad (i = 0, 1, 2, \dots, n-1),$$

$$y(x_1) = y(x_2), \quad y'(x_1) = y'(x_2),$$

and since $J(y)$ contains y' only in the integral this lower limit will be unchanged when the condition $y'(x_1) = y'(x_2)$ is omitted.

The function

$$w = c_0 u_0 + c_1 u_1 + \dots + c_n u_n$$

satisfies the equation $w(x_1) = w(x_2)$ since it vanishes at x_1 and x_2 . Then the $n+1$ constants c_i can always be determined so that they are not all zero, and that the n equations hold:

$$\sum c_k \int_{x_1}^{x_2} (P A y_i u_k dx = 0 \quad (i = 0, 1, 2, \dots, n-1).$$

The common factor of the constants c_i may then be determined so that

$$\int_{x_1}^{x_2} P A w^2 dx = \sum c_k^2 = 1,$$

and the desired inequality follows as in the preceding case.

Suppose y and \bar{y} are functions satisfying the equations

$$(P y')' + (\lambda A - B) P y = 0,$$

$$(P \bar{y}')' + (\bar{\lambda} A - B) P \bar{y} = 0.$$

Then, since $P(x_1) = 1$, $P(x_2) = 1$,

$$[\bar{y} y' - y \bar{y}']_a^b = (\bar{\lambda} - \lambda) \int_a^b A P y \bar{y} dx.$$

Suppose a and b are consecutive zeros of \bar{y} , and that \bar{y} is positive between a and b . Then $\bar{y}'(a) > 0$, $\bar{y}'(b) < 0$, for \bar{y}' cannot vanish with \bar{y} . Suppose further that $\lambda > \bar{\lambda}$ and write $(+)$ for a positive function or constant. Then, since $A > 0$,

$$(+) y(a) + (+) y(b) = - (+) \int_a^b (+) y dx.$$

Hence y must vanish between a and b . That is, between two zeros of a solu-

tion of (1') for a value $\bar{\lambda}$ there lies at least one zero of a solution of (1') for $\lambda > \bar{\lambda}$.*

Suppose now that $\lambda_i = \mu_n$. If y_i and u_n are linearly independent, then y_i does not vanish at x_1 or at x_2 and since by the well known STURM's theorem the zeros of two linearly independent solutions of the same differential equation of type (1') separate each other y_i vanishes $n + 1$ times within the interval (x_1, x_2) . If y_i and u_n are linearly dependent y_i vanishes with u_n , that is, $n + 1$ times in the interval $(x_1 \leq x < x_2)$. But y_i is periodic and therefore $\lambda_i = \mu_n$ only when n is an odd number. Therefore we have from the inequalities $\lambda_n \leq \mu_n \leq \lambda_{n+1}$,

$$\mu_{2m-2} < \lambda_{2m-1} \leq \mu_{2m-1} \leq \lambda_{2m} < \mu_{2m}.$$

If $\lambda_{2m-1} = \mu_{2m-1}$ then y_{2m-1} vanishes $2m$ times in the interval $(x_1 \leq x < x_2)$, as was seen above. If $\lambda_{2m-1} < \mu_{2m-1}$ then, since between each pair of zeros of y_{2m-1} one zero of u_{2m-1} must lie, y_{2m-1} vanishes at most $2m$ times in $(x_1 \leq x < x_2)$. But since $\mu_{2m-2} < \lambda_{2m-1}$, y_{2m-1} vanishes at least $2m - 1$ times in $(x_1 \leq x < x_2)$, and since y_{2m-1} is periodic it follows that y_{2m-1} vanishes in all cases $2m$ times for $x_1 \leq x < x_2$.

If $\lambda_{2m} = \mu_{2m-1}$ then y_{2m} vanishes $2m$ times in $(x_1 \leq x < x_2)$. If $\mu_{2m-1} < \lambda_{2m}$ then y_{2m} has at least $2m$ zeros in $(x_1 \leq x < x_2)$. But since $\lambda_{2m} < \mu_{2m}$, y_{2m} has at most $2m + 1$ zeros in this interval, and since y_{2m} is periodic it follows that it vanishes in all cases exactly $2m$ times for $x_1 \leq x < x_2$. These results may be summarized in the following

THEOREM. *There exists an infinite series of normal parameter values λ_n and corresponding periodic functions y_n with the period $x_2 - x_1$, which satisfy the differential equations*

$$y_n'' + p y_n' + (\lambda_n A - B) y_n = 0,$$

where

$$\int_{x_1}^{x_2} p dx = 0, \quad B \geq 0,$$

and where p , A , B are periodic with the period $x_2 - x_1$. If $A(x) \geq 0$ in the interval (x_1, x_2) the functions y_{2m} and y_{2m-1} vanish exactly $2m$ times for $x_1 \leq x < x_2$.

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* This is a well known result of STURM.